

Perfect Quadrilateral Right Prisms

Allan J. MacLeod,
 Statistics and Mathematics Group,
 University of the West of Scotland,
 High St., Paisley,
 Scotland. PA1 2BE
 (e-mail: allan.macleod@uws.ac.uk)

Abstract

We consider right prisms with horizontal quadrilateral bases and tops, and vertical rectangular sides. We look for examples where all the edges, face diagonals and space diagonals are integers. We find examples when the base is an isosceles trapezium and a parallelogram, but no solution for a kite or rhombus.

1 Introduction

Let **ABCD** be a convex quadrilateral in the x-y plane and let **EFGH** be the identical shape vertically translated h units. Then the 3-dimensional object with vertices **ABCDEFGH** is known as a right prism. In this paper, we consider finding such prisms where the lengths of the edges, face diagonals and space diagonals can all be integers. If **ABCD** and **EFGH** are rectangles, then the prism is in fact a cuboid, and the problem is that of finding a perfect rational cuboid, which is still a classic unsolved problem, see Bremner [1], Dickson [3], Guy [4], Leech[6] and van Luijk [7].

Very recently, Sawyer and Reiter [11] have shown the existence of perfect parallelipeds. These are not normally right prisms, in that the solutions found do not have rectangular vertical faces. In this paper, we consider

quadrilaterals such as the trapezium, the parallelogram, the kite and the rhombus, to see whether perfect right prisms with these base shapes exist. It is interesting to note that a mathematician of the calibre of Kummer [5] considered the problem of quadrilaterals with integer sides and diagonals.

2 Basics

Since the sides of a right prism are clearly rectangles, there will be several equations of the form

$$a^2 + b^2 = \square, \quad a, b \in \mathbb{Z}$$

which have a general solution $a = \alpha(m^2 - n^2), b = \alpha 2mn$. The α term can be a nuisance so we get rid of it by reforming the solution as $a/b = P(r) = (r^2 - 1)/(2r)$ with $r = m/n$.

Often, an equation in a problem reduces to the equation

$$P(c) = d \tag{2.1}$$

where we wish rational solutions. A simple consideration of the underlying quadratic shows that

$$1 + d^2 = \square \tag{2.2}$$

is a necessary condition.

The most usual such occurrence is when we look for rational solutions of

$$x^2 + h^2 = \square, \quad y^2 + h^2 = \square$$

Thus, $x/h = P(a)$ and $y/h = P(b)$ for some rational a and b . Then, $P(b) = (y/x)P(a)$, which has a rational solution if $1 + y^2 P^2(a)/x^2 = \square$, which implies

$$d^2 = y^2 a^4 + (4x^2 - 2y^2)a^2 + y^2 \tag{2.3}$$

has rational solutions. There has clearly at least one rational solution, when $a = 0, d = \pm y$, so the quartic is birationally equivalent to an elliptic curve, see Silverman and Tate [13].

Using standard transformations given in Mordell [8], we find the equivalent curve to be

$$v^2 = u^3 + (x^2 + y^2)u^2 + x^2 y^2 u = u(u + x^2)(u + y^2) \tag{2.4}$$

with $a = v/(y(u + x^2))$.

The curve has torsion subgroup (usually) isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_4$, with finite torsion points of order 2 at $(0, 0)$, $(-x^2, 0)$, $(-y^2, 0)$, and points of order 4 at $(xy, \pm xy(x + y))$, and $(-xy, \pm xy(x - y))$. None of these points give a non-trivial value of $a > 1$. We thus need curves which have rank greater than 0. For example, $x = 2, y = 3$ leads to $v^2 = u^3 + 13u^2 + 36u$ which has rank 0, and hence there are no non-trivial solutions of $4 + h^2 = \square$, $9 + h^2 = \square$.

For $x = 5, y = 2$, however, $v^2 = u^3 + 29u^2 + 100u$ has rank 1 with generator $(2, 18)$. This point gives $a = 1/3$ which is not acceptable. If we add $(2, 18)$ to one of the order 4 torsion points, we get the point $(-20, 40)$, which gives $a = 4$, giving $P(a) = 15/8$ and $h = 8/3$. Scaling x, y, h by 3 gives $x = 15, y = 6, h = 8$.

We look for as many generators as we can easily find. Let G_1, \dots, G_s be the independent generators found with $0 \leq s \leq r$ where r is the rank of the curve, then we compute the points

$$Q = n_1 G_1 + n_2 G_2 + \dots + n_s G_s + T_k \quad (2.5)$$

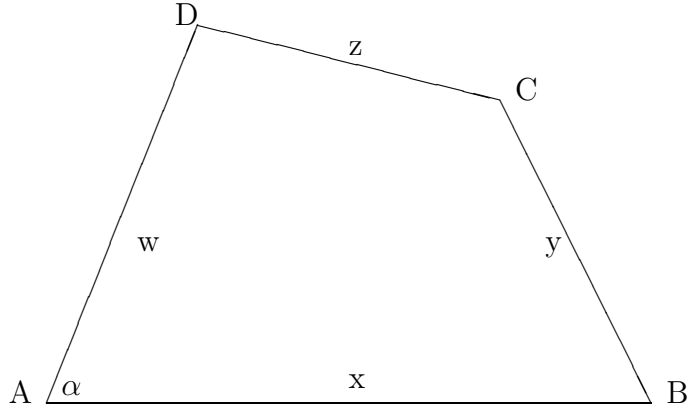
where $-L \leq n_i \leq +L$ and T_k is a torsion point, $0 \leq k \leq 7$, where T_0 is the point at infinity. For each point Q we determine $a, P(a)$ and $h = x/P(a)$.

Suppose P is a point of infinite order. We find P and $P + T$ where T is a point of order 2 generates the 4 values of $\pm a_1, \pm 1/a_1$, for some a_1 , as we use the 3 points of order 2 and the point at infinity. For the four values from $P + T$ where T varies over the 4 points of order 4, we get $\pm a_2, \pm 1/a_2$. It is clear, however, that these values all give the same value for $|P(a_1)|$ and $|P(a_2)|$, and hence we only generate 2 different positive values of h . Further, we have the product of these h values equals $|xy|$. All this can be proven by standard application of the formulae for addition on elliptic curves, using a good symbolic algebra package to alleviate the boredom.

To find whether a curve has strictly positive rank, for a given (x, y) pair, we use the Buhler-Gross-Zagier [2] method of computing the L-series and its derivatives to estimate the rank, if the conductor of the curve is not too large. If the rank is estimated at least 2 or rank=1 and the Birch & Swinnerton-Dyer conjecture predicts a small height for the generator, we search for integer points on the minimal model of the curve, which are then transformed to possible generators of the original curve.

3 General Quadrilateral

This is a convex quadrilateral with no parallel sides and no equal or complementary angles, where we can assume w.l.o.g. that x is the largest side.



We generate values for x and w and $\cos \alpha = m/n$, since integer diagonals imply rational cosines. We first test if $BD = v$ is a rational solution of $v^2 = x^2 + w^2 - 2xw \cos \alpha$. If it is, we generate sides y, z in the range $v < y + z < 2x$ and determine $\cos \angle ABD$, $\sin \angle ABD$, $\cos \angle DBC$ and $\sin \angle DBC$, by elementary trigonometry.

TABLE 1
Near-solution lengths for general quadrilaterals

x	y	z	w	u	v	h	$\cos \alpha$	$\cos B$
91	80	45	63	45	35	60	25/26	113/130
128	56	36	56	88	88	105	23/28	23/28
171	150	84	150	192	192	80	29/100	29/100
159	147	48	147	171	171	140	37/98	37/98
273	112	385	273	343	273	180	1/2	-1/2
441	210	96	210	294	294	280	23/28	23/28
2223	2185	513	2185	2432	2432	420	9/23	9/23
7735	7616	3927	7616	9401	9401	3960	1/4	1/4
23205	16779	5712	16779	20349	20349	2860	49/94	49/94

These relations give $\cos B$, and we test the rationality of $u = AC$ from $u^2 = x^2 + y^2 - 2xy \cos B$. If we find rational u and v , we test for

$$\{x, y, z, w, u, v\}^2 + h^2 = \square$$

by using the elliptic curve method of section 2. We find, if possible, values of h with $\{x^2, y^2\} + h^2 = \square$ and test these with $\{z, w, u, v\}$.

Despite extensive testing we were unable to find a perfect prism but we found several where only one of the side or space diagonals is not an integer. These are given in Table 1. A quick survey of Table 1 shows that all the quadrilaterals have at least one pair of lengths identical. We have been unable to find any quadrilaterals, with all horizontal lengths distinct, where more than 4 vertical diagonals are rational. The 5 results found so far are given in Table 2.

TABLE 2
Distinct-sided quadrilaterals with 4 rational vertical diagonals.

x	y	z	w	u	v	h	$\cos \alpha$	$\cos B$
152	140	56	36	88	154	105	1/16	23/28
390	380	315	95	374	425	168	-5/19	251/475
420	60	600	288	472	540	175	-2/15	-191/225
728	504	462	266	640	735	2520	5/32	25/49
1672	616	396	1540	1664	968	1155	23/28	167/847

4 Isosceles Trapezium

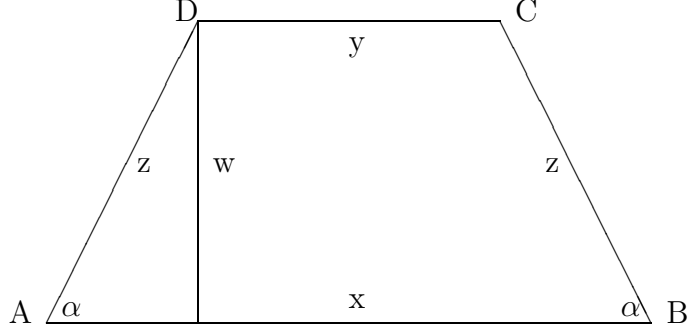
An isosceles trapezium is a shape as shown in the diagram, where AB and DC are parallel, and $x, y, z \in \mathbb{Z}$. Thus $\cos \alpha = (x - y)/2z$, so that $\cos \alpha$ is rational.

The perpendicular gap w satisfies

$$w^2 = z^2 - (x - y)^2/4 \tag{4.1}$$

and the base diagonal is $v = AC = BD$, which satisfies

$$v^2 = \frac{(x + y)^2}{4} + w^2 = z^2 + xy \tag{4.2}$$



The vertical side diagonals give the three equations

$$x^2 + h^2 = \square \quad , \quad y^2 + h^2 = \square \quad , \quad z^2 + h^2 = \square \quad (4.3)$$

whilst there is only one space diagonal equation, namely $v^2 + h^2 = \square$.

We have $x/h = P(a)$, $y/h = P(b)$, $z/h = P(c)$ for some rational $a, b, c > 1$. We, thus, generate values of a, b, c , find $P(a), P(b), P(c)$ and hence suitable integer x, y, z, h . We reject those for which, firstly $z^2 + xy \neq \square$ and then $v^2 + h^2 \neq \square$.

This simple search was coded in Pari-GP and quickly found several solutions. The 10 smallest (using the measure $\|\cdot\| = x + y + z + v + h$) are given in Table 3.

TABLE 3
Solution lengths for isosceles trapezia

x	y	z	v	h
364	275	320	450	240
1152	507	780	1092	1040
3325	1053	1620	2475	2160
3328	361	1824	2128	3420
3549	2601	1326	3315	1768
4225	2527	2223	3952	2964
5632	4693	1368	5320	1824
2754	1984	4455	5031	7560
6647	3168	5950	7514	2040
10633	2300	4550	6720	5040

We can parameterise the lengths of the isosceles trapezium by parameterising $v^2 = x^2 + z^2 - 2xz \cos \alpha$. Standard algebra leads to the fomulae,

$$x = 2(c + d) \quad , \quad y = 2d(1 + cd) \quad , \quad z = 1 - d^2 \quad , \quad v = 2cd + 1 + d^2 \quad (4.4)$$

where $c, d < 1$, and, in fact, $c = \cos \alpha \equiv i/j$ with $i, j \in \mathbb{Z}$.

The expression for z leads to the temptation to define $h = 2d$ so that $z^2 + h^2 = (1 + d^2)^2$. Given this h , for $x^2 + h^2$ and $y^2 + h^2$ to be squares, we need, respectively

$$c^2 + 2cd + 2d^2 = \square \quad , \quad c^2 d^2 + 2cd + 2 = \square \quad (4.5)$$

while $v^2 + h^2$ square needs

$$d^4 + 4cd^3 + (4c^2 + 6)d^2 + 4cd + 1 = \square \quad (4.6)$$

The quadric $c^2 + 2cd + 2d^2 = \square$ can be easily parameterised by $c = g^2 - 2f^2$ and $d = 2f(f - g)$, but when we substitute these into the second quadric we get a sextic in g or an octic in f which must be made square. These curves are likely to have genus greater than 1 and so only a finite number of rational solutions, by Mordell's theorem.

The latter quartic clearly is birationally equivalent to an elliptic curve and we find this curve to be

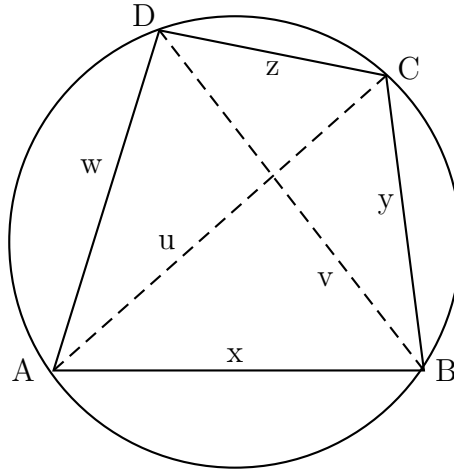
$$v^2 = u^3 + i^2 u^2 - j^4 u \quad , \quad d = \frac{v - iu}{j(u + j^2)} \quad (4.7)$$

This curve seems, from numerical experiments just to have the single finite torsion point $(0, 0)$. It has the rational points $(j^2, \pm i j^2)$ and $(-j^2, \pm i j^2)$. If we double the point $(j^2, i j^2)$ we get $(j^4/i^2, -j^6/i^3)$ and since we assume i/j is a completely reduced fraction, we have non-integer coordinates, so by the Nagell-Lutz theorem this is of infinite order.

Using $u = j^4/i^2, v = j^6/i^3$ gives $d = (j^3 - i^2 j)/(i^3 + i j^2)$ which means that $x^2 + h^2 = \square$ when $i^8 + 3i^4 j^4 - 2i^2 j^6 + 2j^8 = \square$, whilst $y^2 + h^2$ is square when $i^4 + 2i^2 j^2 + 5j^4 = \square$. The latter is birationally equivalent to the elliptic curve $v^2 = u^3 - u^2 - u$ which has rank 0, so this choice of point does not lead to all vertical diagonals square.

5 Cyclic Quadrilateral

The isosceles trapezium is an example of a cyclic quadrilateral - a quadrilateral whose 4 vertices lie on the circumference of a circle.



We look for lengths (x, y, z, w, u, v) which are all positive integers first. This has been considered by Schubert [12] and Sastry [9] [10]. There is an error in the basic formulae in [9], whilst [10] uses a somewhat unusual parameterisation.

We derive a more standard form here. The results are based on the well-known formula for the length of an arc of a circle of radius R , namely $2R\sin\theta$ where θ is the angle the arc makes with the circumference. Let $\alpha = \angle ADB = \angle ACB$, $\beta = \angle BAC = \angle BDC$, $\gamma = \angle CAD = \angle CBD$ and $\delta = \angle ABD = \angle ACD$. Since ABCD is a cyclic quadrilateral, opposite angles are complimentary, thus $\alpha + \beta + \gamma + \delta = \pi$, so there are only 3 free choices of angle.

Therefore, assuming w.l.o.g. than $2R = 1$, we have

$$x = \sin \alpha \quad , \quad y = \sin \beta \quad , \quad z = \sin \gamma \quad (5.1)$$

$$w = \sin \alpha \cos \beta \cos \gamma + \cos \alpha \sin \beta \cos \gamma + \cos \alpha \cos \beta \sin \gamma - \sin \alpha \sin \beta \sin \gamma \quad (5.2)$$

$$u = \sin \alpha \cos \beta + \cos \alpha \sin \beta, \quad v = \sin \beta \cos \gamma + \cos \beta \sin \gamma \quad (5.3)$$

Thus, the sines and cosines of these angles are rational so we can assume

$$\begin{aligned} \cos \alpha &= \frac{f^2 - 1}{1 + f^2}, \quad \sin \alpha = \frac{2f}{1 + f^2}, \quad \cos \beta = \frac{g^2 - 1}{1 + g^2}, \quad \sin \beta = \frac{2g}{1 + g^2}, \\ \cos \gamma &= \frac{h^2 - 1}{1 + h^2}, \quad \sin \gamma = \frac{2h}{1 + h^2}, \end{aligned} \quad (5.4)$$

with $f, g, h \in \mathbb{Q}$.

Clearing denominators, we have

$$x = f(g^2 + 1)(h^2 + 1), \quad y = g(f^2 + 1)(h^2 + 1) \quad (5.5)$$

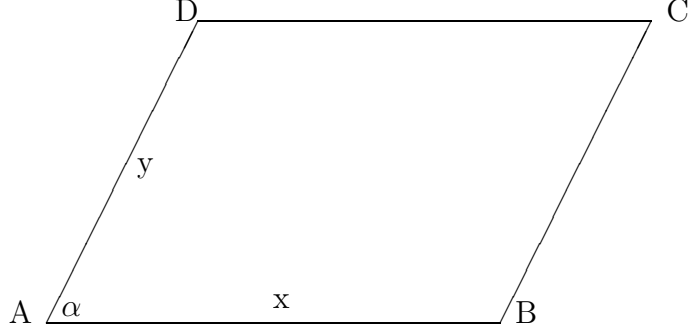
$$z = h(f^2 + 1)(g^2 + 1), \quad w = (f(g + h) + gh - 1)(f(gh - 1) - g - h) \quad (5.6)$$

$$u = (f + g)(fg - 1)(1 + h^2), \quad v = (g + h)(gh - 1)(1 + f^2) \quad (5.7)$$

Despite extensive computing using these formulae, we have been unable to generate a cyclic quadrilateral, with all sides and diagonals mutually distinct, where more than 3 of the $t^2 + h^2$ terms are square. The smallest quadrilateral with 3 squared terms found so far is $x = 561$, $y = 750$, $z = 1275$, $w = 1560$, $u = 1575$, $v = 1197$, $h = 400$, $\cos \alpha = 11/2$, $\cos \beta = 4$ and $\cos \gamma = 2$.

6 Parallelogram

The parallelogram $ABCD$ has sides $AB \parallel DC$ and $AD \parallel BC$, with $AB = DC = x$ and $AD = BC = y$.



Let $AC = z$ and $BD = w$. Then

$$z^2 = (x + y \cos \alpha)^2 + (y \sin \alpha)^2 = x^2 + y^2 + 2xy \cos \alpha \quad (6.1)$$

$$w^2 = (x - y \cos \alpha)^2 + (y \sin \alpha)^2 = x^2 + y^2 - 2xy \cos \alpha \quad (6.2)$$

These are both homogeneous quadrics, and the intersection of two such elements is known to be an elliptic curve. We can parameterise the first equation as

$$\frac{x}{y} = \frac{2(\cos \alpha - \mu)}{\mu^2 - 1}$$

which we substitute into the second quadric giving

$$d^2 = \mu^4 + 4k\mu^3 + 2(1 - 2k^2)\mu^2 - 12k\mu + 8k^2 + 1 \quad (6.3)$$

where $k = \cos \alpha = m/n$ since $\cos \alpha$ clearly rational.

This quartic is birationally equivalent to an elliptic curve, which we eventually find to be

$$h^2 = g^3 + 2(m^2 + n^2)g^2 + (m^2 - n^2)^2g \quad (6.4)$$

with the relation

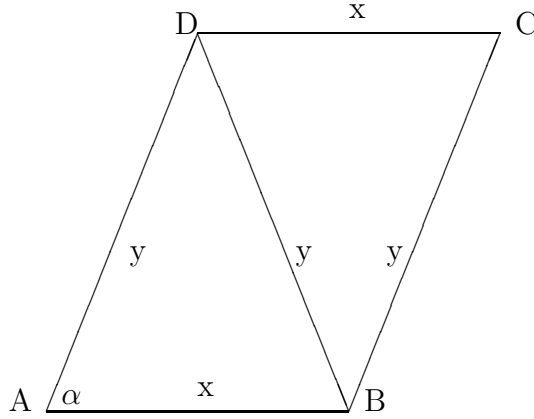
$$\mu = \frac{h - 2m(m^2 - n^2)}{n(g - (m^2 - n^2))} \quad (6.5)$$

The elliptic curve has torsion subgroup isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_4$, with finite torsion points of order 2 for $g = 0, -(m + n)^2, -(m - n)^2$ and order 4 points at $g = \pm(m^2 - n^2)$. None of these lead to non-trivial μ and x/y . Thus we need curves of strictly positive rank. We use the same approach as described

in section 2 to find curves with infinite generators. We generate multiples of such generators and these give different μ and hence different x, y, z, w .

For given x, y values, we use the method of section 2 to find values of h , if any, where $x^2 + h^2 = \square$ and $y^2 + h^2 = \square$. These values of h are then tested for $z^2 + h^2 = \square$ and $w^2 + h^2 = \square$.

Unfortunately, despite extensive testing, no solutions were found. In fact, only rarely did we have one of $z^2 + h^2$ or $w^2 + h^2$ a square, and this almost always occurred when z or w equalled x or y .



Consider the special parallelogram $ABCD$ given above. Then $\cos \alpha = x/(2y)$ and $AC^2 = z^2 = 2x^2 + y^2$. This can be simply parameterized by $x = 2pq, y = p^2 - 2q^2, z = p^2 + 2q^2$. We can thus easily generate values of x, y, z and hence use section 2 to find values of h .

A large amount of testing has found a single solution where all sides, face diagonals and space diagonals are integers. This occurs where $x = 2522, y = 16199$, giving $w = 16199, z = 16587$, and $\cos \alpha = 13/167$. If we have a height $h = 9240$, we have side diagonals of 9578 and 18649, and the two space diagonals 18649 and 18987.

It might be thought that we can set $h = p^2 - q^2$ since $x^2 + h^2 = \square$. For $y^2 + h^2$ and $z^2 + h^2$ to be square we need, respectively, $2p^4 - 6p^2q^2 + 5q^4 = \square$ and $2p^4 + 2p^2q^2 + 5q^4 = \square$. Both have obvious solutions, so are birationally equivalent to elliptic curves. We find the curves and transformations to be

$$E_1 : v^2 = u^3 + 12u^2 - 4u \quad , \quad \frac{p}{q} = \frac{4u + v + 4}{2u + v - 4} \quad (6.6)$$

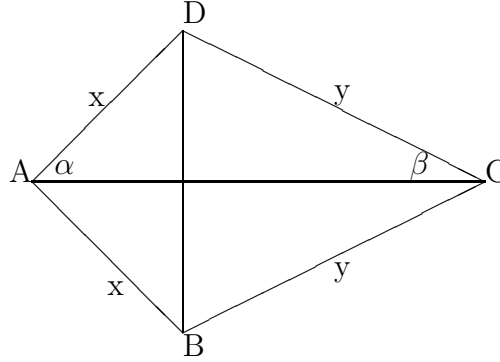
$$E_2 : v^2 = u^3 - 4u^2 - 36u \quad , \quad \frac{p}{q} = \frac{4u + v + 12}{v - 2u - 12} \quad (6.7)$$

in the correct order.

Both curves have only one finite torsion point $(0, 0)$, both have rank 1 with E_1 having generator $G_1 = (1, 3)$ and E_2 generator $G_2 = (9, 9)$. We can easily generate, using Pari-GP, the points on E_1 , compute p/q and see if the second quartic is square. We have done this up to $(u, v) = 99G_1$ with no luck.

7 Kite

A standard kite has the shape shown where the diagonals meet at right-angles with BD bisected by AC . Call $\angle DAC = \alpha$ and $\angle DCA = \beta$.



Thus $AC = z = x \cos \alpha + y \cos \beta$ and $BD = w = 2x \sin \alpha = 2y \sin \beta$. Thus we must have the sines and cosines rational, so we assume

$$\cos \alpha = \frac{f^2 - 1}{f^2 + 1}, \quad \sin \alpha = \frac{2f}{kf^2 + 1}, \quad \cos \beta = \frac{g^2 - 1}{g^2 + 1}, \quad \sin \beta = \frac{2g}{g^2 + 1},$$

where $f, g \in \mathbb{Q}$ and $f, g > 1$.

The equation for w gives

$$g^2 - 2kg + 1 = 0 \quad , \quad k = \frac{y}{x \sin \alpha} \quad (7.1)$$

so that a rational g needs $k^2 - 1 = \square$, and $g = k + \sqrt{k^2 - 1}$.

We first find x and y such that the elliptic curve from section 2 has positive rank. We then generated values of $f = m/n$ with $m + n \leq 299$, and tested for rational g . If such g existed, we computed z , w and h and tested for $z^2 + h^2 = \square$ and $w^2 + h^2 = \square$. In the testing so far, no example with both conditions satisfied has been found. Many examples with one satisfaction were found, the vast majority with $z = x$ or $z = y$. Table 4 lists some examples where all the lengths are different.

TABLE 4
Nearly-perfect kites

x	y	z	w	h	$\cos \alpha$	$\cos \beta$
75	435	450	144	308	7/25	143/145
100	208	252	160	105	3/5	12/13
585	1190	1375	1008	1200	33/65	77/85
3300	13260	13800	6336	9625	7/25	1073/1105

It should be noted that it is easy to see that we cannot have $w = x$ or $w = y$. We can simply parameterise a kite as having sides and diagonals which are multiples of

$$x = g(f^2 + 1) \text{ , } y = f(g^2 + 1) \text{ , } z = f(g^2 - 1) + g(f^2 - 1) \text{ , } w = 4fg \quad (7.2)$$

If we define $h = 4f^2 - g^2$ then $w^2 + h^2 = \square$. To have $x^2 + h^2 = \square$, we must have

$$(g^2 + 16)f^4 - 6g^2f^2 + g^2(g^2 + 1) = \square \quad (7.3)$$

and a little searching shows that $f = g/2$ gives a solution. Thus this quartic is birationally equivalent to an elliptic curve.

Standard analysis shows that the equivalent curve is

$$v^2 = u^3 + 12p^2q^2u^2 - 4p^2q^2(p^2 + 4q^2)^2u \quad (7.4)$$

where $g = p/q$ with p, q integers with no common factors. Numerical investigations suggest that the obvious point $(0, 0)$ is the only common torsion point, and that, in general, the torsion subgroup is isomorphic to \mathbb{Z}_2 .

The transformation back from the elliptic curve is

$$f = \frac{p(v + 4q^2u + 8p^2q^2(p^2 + 4q^2))}{2q(v - p^2u - 8p^2q^2(p^2 + 4q^2))} \quad (7.5)$$

If we set the denominator to 0, we get $v = p^2(u + 8p^2q^2 + 32q^4)$, and if we substitute into the elliptic curve, we find $u = -16p^2q^2$, $v = \pm 8p^2q^2(p^2 - 4q^2)$. Since we suspect that there is only one finite torsion point, this would have to be a point of infinite order. Doubling this point gives

$$u = \frac{(p^4 + 72p^2q^2 + 16q^4)^2}{16(p^2 - 4q^2)^2}$$

and the fractional nature of this shows that $-16p^2q^2$ will give a point of infinite order for most (p, q) pairs. Thus, the elliptic curve has rank at least 1 in most cases, with the largest rank found so far being 4, starting with $p = 17, q = 8$. So, we can easily find kites with $(x^2, w^2) + h^2 = \square$.

We wrote a simple program which searched for other generators on these curves, generated points on the curve, and hence (x, y, z, w, h) sets which we checked for more than two $a^2 + h^2 = \square$. These were very sparsely distributed and we only found one new solution with 3 squares - $(x, y, z, w, h) = (8745, 4182, 10881, 6336, 14840)$.

It might be thought that making $y^2 + h^2 = \square$ first would produce a similar outcome. We require,

$$16f^4 + (g^4 - 6g^2 + 1)f^2 + g^4 = \square \quad (7.6)$$

which can be linked to the elliptic curve

$$v^2 = u^3 - 2(p^4 - 6p^2q^2 + q^4)u^2 + (p^2 + q^2)^2(p^2 + 4pq + q^2)(p^2 - 4pq + q^2)u \quad (7.7)$$

with $g = p/q$ and transformation $f = v/(8q^2u)$.

These curves can have rank 0 and have torsion subgroup isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_4$. Using this form, we found no perfect kites and only one new near-perfect kite with sides $(1883, 1924, 1107, 3640, 2400)$.

Finally, we can look to make $(w^2, z^2) + h^2 = \square$. Using the prior parameterisations, we require

$$(g^2 + 16)f^4 + 2g(g^2 - 1)f^3 + (g^4 - 12g^2 + 1)f^2 - 2g(g^2 - 1)f + g^2(g^2 + 1) = \square \quad (7.8)$$

where, again, we find $f = g/2$ gives a solution. Transforming, we find the elliptic curve

$$v^2 = u^3 + (p^4 + 18p^2q^2 + q^4)u^2 + 12p^2q^2(p^4 - 4q^4)u \quad (7.9)$$

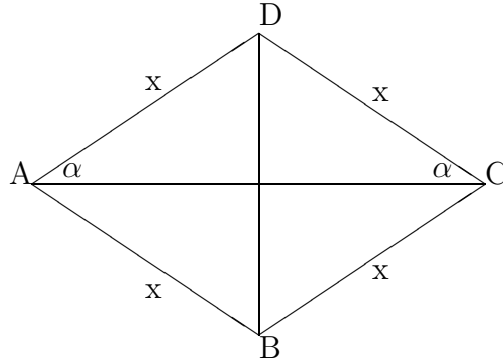
where $g = p/q$ and the inverse transformation is

$$f = \frac{p(v + (p^2 - 5q^2)u + 24p^2q^2(p^2 - 2q^2))}{2q(v + (q^2 - 2p^2)u - 24p^2q^2(p^2 - 2q^2))} \quad (7.10)$$

As before, the torsion subgroup seems to only contain $(0, 0)$ as a finite point. Solving either the numerator or denominator for v and substituting into the elliptic curve gives the point $(-16p^2q^2, \pm 8p^2q^2(p^2 + 4q^2))$, so the curves have rank at least 1. Numerical experiments show that these curves can have quite a large rank - the largest found is 6 starting with $p = 29, q = 15$ - and the rank was always at least 2. This led to investigating the simple integer points and the discovery of a second point of infinite order $(-(p^2 + q^2)^2, \pm 2pq(p^2 + q^2)(p^2 + 4q^2))$. Surprisingly, given all points available, we were unable to find any kites with even 3 vertical diagonals integer.

8 Rhombus

A rhombus is both a kite and a parallelogram with shape



Let $AC = z = 2x \cos \alpha$ and $BD = w = 2x \sin \alpha$, so $\cos \alpha$ and $\sin \alpha$ must be rational so set

$$\cos \alpha = \frac{f^2 - 1}{f^2 + 1} \quad , \quad \sin \alpha = \frac{2f}{f^2 + 1}$$

where $f = m/n > 1$.

We thus need

$$x^2 + h^2 = \square \quad , \quad \frac{4(m^2 - n^2)^2 x^2}{(m^2 + n^2)^2} + h^2 = \square \quad , \quad \frac{16m^2 n^2 x^2}{(m^2 + n^2)^2} + h^2 = \square \quad (8.1)$$

and we can determine the simple parameterisation

$$x = m^2 + n^2 \quad , \quad z = 2(m^2 - n^2) \quad , \quad w = 4mn \quad (8.2)$$

If we set $x = p^2 - q^2$ and $h = 2pq$, we have $x^2 + h^2 = \square$. Thus we need $m^2 + n^2 + q^2 = p^2$. One parameterisation is $m = 2ab$, $n = 2ac$, $q = a^2 - b^2 - c^2$, $p = a^2 + b^2 + c^2$. Substituting, we find $z^2 + h^2 = \square$ requires

$$4(a^8 + 2(7b^4 - 18b^2c^2 + 7c^4)a^4 + (b^2 + c^2)^2) = \square \quad (8.3)$$

whilst $w^2 + h^2 = \square$ needs

$$4(a^8 - 2(b^4 - 30b^2c^2 + c^4)a^4 + (b^2 + c^2)^2) = \square \quad (8.4)$$

It might be thought, at first, that this is an octic expression and therefore fairly unmanageable. Note, however, that $x = 4a^2(b^2 + c^2)$, $z = 8a^2(b^2 - c^2)$, $w = 16a^2bc$, so the term of interest is actually a^2 , which makes these expressions quartics and hence equivalent to elliptic curves.

We find that the expression for $z^2 + h^2$ is related to the curve

$$v^2 = w^3 - (7b^4 - 18b^2c^2 + 7c^4)w^2 + 4(b^2 - c^2)^2(b^2 - 3c^2)(3b^2 - c^2)w \quad (8.5)$$

and $w^2 + h^2$ related to

$$v^2 = w^2 + (b^4 - 30b^2c^2 + c^4)w^2 - 16b^2c^2(b^4 - 14b^2c^2 + c^4)w \quad (8.6)$$

with, in both cases, $a^2 = v/w$. Both curves have at least 8 torsion points, with, for most (b, c) pairs, the torsion subgroup isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_4$.

Extensive testing using both elliptic curves to generate values of a^2 from input (b, c) pairs has not found any examples of a perfect right rhombus prism. For $x^2, z^2 + h^2 = \square$ a simple example is $x = 75, z = 42, w = 144, h = 40$, and , for $x^2, w^2 + h^2 = \square$ a simple example is $x = 988, z = 760, w = 1824, h = 315$.

References

- [1] A. Bremner, *The rational cuboid and a quartic surface*, Rocky Mtn. J. Math. **18**, (1988), pp 105 – 121.
- [2] J. Buhler, B. Gross, and D. Zagier, *On the conjecture of Birch and Swinnerton-Dyer for an elliptic curve of rank 3*, Math. Comp. **44**, (1985), pp 473 – 481
- [3] L.E. Dickson, *History of the Theory of Numbers, Volume II, Diophantine Analysis*, AMS Publishing, New York, 1992.
- [4] R.K. Guy, *Unsolved Problems in Number Theory*, Springer, New York, 2005.
- [5] E.E. Kummer, *Über die Vierecke, deren Seiten und Diagonalen rational sind*, Journal für die Reine Ang. Math. **37**, (1848), pp 1 – 20.
- [6] J. Leech, *The rational cuboid revisited*, Amer. Math. Monthly **84**, (1977), pp 518 – 533.
- [7] R. van Luijk, *On Perfect Cuboids*, Undergraduate thesis, University of Utrecht, (2000).
- [8] L.J. Mordell, *Diophantine Equations*, Academic Press, New York, 1968.
- [9] K.R.S. Sastry, *Brahmagupta quadrilaterals*, Forum Geom. **2**, (2002), pp 167 – 173.
- [10] K.R.S. Sastry, *Brahmagupta quadrilaterals: a description*, Crux Math. **29**, (2003), pp 39 – 42.
- [11] J.F. Sawyer and C.A. Reiter, *Perfect Parallelepipeds Exist*, Mathematics of Computation, to appear.
- [12] H. Schubert, *Die Ganzzahligkeit in der Algebraischen Geometrie*, Leipzig, 1905.
- [13] J.H. Silverman and J. Tate, *Rational Points on Elliptic Curves*, Springer, Undergraduate Texts in Mathematics, New York, 1994.